Piecewise Linear Yield Criteria in the Problems of Thermoplasticity

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***Abstract*—The present study is devoted to the boundary value problems of the perfect thermoelastic-plastic continua concerning to the hollow cylinder deforming under non-stationary thermal action. The conventional Prandtl-Reuss elastic-plastic model generalized on the thermal effects is used. The irreversible deformations, residual stresses, and displacements are computed due to analytical solutions within the framework of the piecewise linear Tresca and Ishlinsky-Ivlev yield criteria. It is shown the possibility to estimate the accuracy of numerical solution obtaining within frameworks of the von Mises yield criterion. The residual stresses of a hollow elastic-thermoplastic cylinder processing by unsteady thermal action on the inner surface are calculated. A new technique for verifying of the numerical solutions accuracy in problems of the theory of perfect plasticity is proposed.**

***Index Terms*—elasticity, heat conduction, Ishlinsky–Ivlev yield criterion, Tresca yield criterion, von Mises yield criterion, maximum reduced stress, plasticity, residual strain, thermal stress.**

I. Preliminary Remarks

One of the problems of the irreversible deformation mechanics is the calculation of the stress-strain state parameters during uneven heating–cooling. On this way it is possible to estimate the residual stresses values. The high residual stresses values forming during an elastic-plastic material cooling lead to material fracture. The residual stresses are the essential factor in most technological processes including natural phenomena and additive manufacturing. The problems concerning residual stresses computations in the frameworks of the large elastic-plastic deformations are discussed in [1], [2], [3]. Some results were presented in studies concerning residual stress calculations within the frameworks of the surface growth theory to problems in geomechanics (e.g., see [4]) and additive manufacturing technologies (e.g., see [5], [6], [7], [8], [9]). Thus, the prediction of the residual stresses values under different heat processing regimes is an actual problem of the modern continuum mechanics. Studies within thermal stresses theory frameworks show that the residual stresses is in proportion to the irreversible deformations. At present, there are many analytical solutions to one-dimensional problems obtained by the Tresca yield criterion [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. The equilibrium equations integrating in terms of the displacement vector give us the analytical solution in the framework of the perfect plasticity theory. The Tresca yield criterion accurately describes the elastic-plastic behaviour of a material under shear deformations. Another piecewise linear yield criterion is the Ishlinsky-Ivlev (maximum reduced stress) yield criterion (see in details [14], [15], [20], [21]). The von Mises yield criterion is more preferred in the case of thermal expansion. The nonlinearity of this condition leads to the necessity of numerical integration of the equilibrium equations even in the one-dimensional case. Numerical algorithms have a high error value for non-stationary nonuniform temperature effects calculation in consequence of the simultaneous existence of plastic flow and unloading domains and elastic-plastic boundaries motion. At present study we proposed a technique of the residual stress computing in the frameworks of the piecewise linear Tresca and Ishlinsky-Ivlev yield criteria. In the stationary thermal action case similar solutions correspond to the von Mises yield criterion ones. As shown below, the calculation process within frameworks of the piecewise linear yield criterion is simpler and faster in the non-stationary case in contrary to the numerical integration by virtue of the von Mises yield criterion.

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II. Problem Statement and Governing Equations

Let consider the hollow cylinder with inner and outer radii R_1 and R_2 respectively.

We assume that the isotropic elastic-plastic material of the cylinder obeys the conventional Prandtl–Reuss model [14], [22]. The infinitesimal strains d_{ij} are separated by the elastic (reversible) e_{ij} and the plastic (irreversible) p_{ij} deformations. Thus, in cylindrically symmetric case the following equations are derived

$$\begin{gathered} d_{rr} = u_{r,r} = e_{rr} + p_{rr}, \\ d_{\varphi\varphi} = \frac{u_r}{r} = e_{\varphi\varphi} + p_{\varphi\varphi}, \\ d_{zz} = p_{zz} + e_{zz}. \end{gathered} \tag{1}$$

u_r is radial component of the displacement vector. The index after comma denotes the partial derivative with respect to corresponding spatial coordinate.

The face surfaces of the cylinder are fixed:

$$d_{zz} = 0. \tag{2}$$

The lateral surfaces are loads free:

$$\sigma_{rr}(R_1, t) = 0, \quad \sigma_{rr}(R_2, t) = 0. \tag{3}$$

The level and distribution of elastic deformations and the temperature field inside the plate give us the stresses obeying the Duhamel–Neumann law

$$\begin{aligned} \sigma_{rr} &= (\lambda + 2\mu)e_{rr} + \lambda(e_{\varphi\varphi} + e_{zz}) - (3\lambda + 2\mu)\Delta, \\ \sigma_{\varphi\varphi} &= (\lambda + 2\mu)e_{\varphi\varphi} + \lambda(e_{rr} + e_{zz}) - (3\lambda + 2\mu)\Delta, \\ \sigma_{zz} &= (\lambda + 2\mu)e_{zz} + \lambda(e_{\varphi\varphi} + e_{rr}) - (3\lambda + 2\mu)\Delta. \end{aligned} \tag{4}$$

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where $\Delta(r,t)=\alpha(T(r,t)-T_0)$ is the thermal expansion of the cylinder being proportionate to the difference between actual and referential temperatures; α is the coefficient of linear thermal expansion; λ, μ are the Lame modulus.

The thermal stresses inside the cylinder should satisfy to equilibrium equation

$$\sigma_{rr,r}+\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{r}=0, \tag{5}$$

Temperature field can be obtained by integrating of the heat conduction equation under given boundary conditions. We assume that the temperature of the outer cylindrical surface is given constant T_0, and the temperature of the inner cylindrical surface depends on time t :

$$T(r,t)-T_0=\frac{t\ln(r/R_2)}{\ln(R_1/R_2)}. \tag{6}$$

The thermal stresses are changed in virtue of the gradual increasing of the temperature gradient. It becomes possible the irreversible deformations accumulation. Plastic flow process is coupled with the yield criterion satisfaction. The following three yield criteria are widely used in solid mechanics: piecewise linear Tresca yield criterion [10] (maximum tangential stress one)

$$f=\max\{|\sigma_1-\sigma_2|,|\sigma_2-\sigma_3|,|\sigma_3-\sigma_1|\}-2k=0; \tag{7}$$

piecewise linear Ishlinsky–Ivlev yield criterion [14], [15] (maximum reduced stress one)

$$f=\max\{|\sigma_1-\sigma|,|\sigma_2-\sigma|,|\sigma_3-\sigma|\}-\frac{4k}{3}=0; \tag{8}$$

von Mises yield criterion [23], [24] (maximum equivalent tensile stress one)

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=8k^2. \tag{9}$$

Herein, $\sigma=(\sigma_1+\sigma_2+\sigma_3)/3$, $\sigma_1=\sigma_{rr}$, $\sigma_2=\sigma_{\varphi\varphi}$, $\sigma_3=\sigma_{zz}$. We assume that the yield stress is the linear function of the actual temperature

$$k(T)=k_0(1-\beta(T-T_0)), \tag{10}$$

wherein k_0 is the referential yield stress, β is the constitutive constant, which can be experimentally obtained.

The yield creteria (7)–(9) can be interpreted as some surface manifesting plastic properties of the solids in the Haigh-Westergaard stress space. In particular, the Tresca and Ishlinsky-Ivlev yield criteria within frameworks of the HaighWestergaard stress space are presented as the hexagonal prisms inclined to the coordinate axes, and the von Mises one is the cylinder. The projections of the Tresca and the Ishlinsky-Ivlev yield criteria on deviatoric plane shown on Fig. 2 are the regular hexagons with a center lying on the hydrostatic axis, and the similar projection of von Mises yield criteria is the circle of radius $2\sqrt{2/3}k$.

The yield criterion is stated the plastic potential due to von Mises maximum principle. That implicit the associated flow rule as the general constitutive equation of the flow theory

$$dp_{ij}=d\xi\frac{\partial f}{\partial\sigma_{ij}},\quad dp_{ij}=d\xi_1\frac{\partial f_1}{\partial\sigma_{ij}}+d\xi_2\frac{\partial f_2}{\partial\sigma_{ij}}. \tag{11}$$

The right equations (11) correspond to the edge of the piecewise linear yield criteria (7), (8).

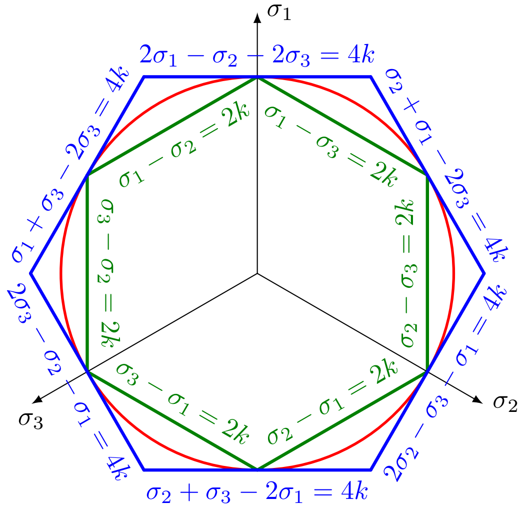

Fig. 1. Yield criteria in deviatory plane. The red circle is the von Mises yield criterion. The inscribed green hexagon is the Tresca yield criterion. The escribed blue hexagon is the Ishlinsky-Ivlev yield criterion. σ_i' are the projections of the principal stresses at deviatory plane.

III. Plastic Flow

A. Tresca yield condition

The time of the plastic flow arising can be determined by an yield criterion. For considered boundary value problem the plastic flow is begun at inner hollow cylinder surfaces. The plastic flow domain for Tresca yield criterion consist of the two subdomains:

1) complete plasticity subdomain $R_1<r<a_2$ satisfying the Tresca prism edge equations

$$\begin{gathered}\sigma_{rr}-\sigma_{\varphi\varphi}=2k,\\ \sigma_{rr}-\sigma_{zz}=2k,\\ p_{rr}+p_{\varphi\varphi}+p_{zz}=0,\end{gathered} \tag{12}$$

2) plastic flow domain $a_2<r<a_1$ corresponding to the Tresca prism facet equations

$$\begin{gathered}\sigma_{rr}-\sigma_{zz}=2k,\\ p_{rr}+p_{zz}=0,\\ p_{\varphi\varphi}=0.\end{gathered} \tag{13}$$

The plastic deformations are obtained by the equations (1), (4), (5), (12), (13) for domain $R_1<r<a_1$:

$$\begin{gathered}p_{rr}^{(1)}=\frac{1}{q}\left(G_1(r,t)-G_{-1}(r,t)+\frac{c_1(t)}{2}\right)-\\ -\frac{2k(r,t)}{\gamma}-3F_1(r,t)+2\Delta(r,t)-\frac{c_2(t)}{r^2},\\ p_{zz}^{(1)}=\frac{2G_{-1}(r,t)}{q}-\frac{c_1(t)}{q}+\frac{k(r,t)}{\gamma}-\Delta(r,t),\\ G_m(r,t)=\frac{1}{r^{(m+1)}}\int_{R_1}^{r}k(\rho,t)\rho^m d\rho,\\ F_m(r,t)=\frac{1}{r^{(m+1)}}\int_{R_1}^{r}\Delta(\rho,t)\rho^m d\rho,\\ q=(3\lambda+2\mu),\quad \gamma=\frac{\mu q}{(\lambda+\mu)},\end{gathered} \tag{14}$$

for domain $a_1 < r < a_2$:

$$
\begin{aligned}
p_{rr}^{(2)} &= \frac{\eta(G_\eta(r,t) - G_{-\eta}(r,t))}{4(\lambda+\mu)} + \frac{\gamma\Delta(r,t)}{2\mu} + \\
&+ \frac{\gamma}{4\mu}\left(F_{-\eta}(r,t) - (\eta+1)F_\eta(r,t)\right) - \frac{\eta^2 k(r,t)}{2\mu} + \\
&+ \frac{\eta r^{(\eta-1)} c_3(t)}{2} - \frac{\eta c_4(t)}{2r^{(\eta+1)}}, \quad \eta = \sqrt{\frac{\lambda+2\mu}{\lambda+\mu}}.
\end{aligned}
\tag{15}
$$

The unknown functions $c_i(t)$ presented in the eqs. (14), (15) are found from the boundary conditions (3) and continuity conditions for the deformations (1) at the elastic-plastic borders a_i. The position of the elastic-plastic borders is computed by the system of equations: $p_{rr}^{(1)}(a_1,t) = p_{zz}^{(1)}(a_1,t)$, $p_{rr}^{(2)}(a_2,t) = 0$.

B. Ishlinski-Ivlev yield condition

The irreversible deformation corresponding to the Ishlinsky-Ivlev yield criterion is divided on the three subdomains discriminating by following conditions

1) subdomain $R_1 < r < b_1$

$$
\begin{gathered}
2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{zz} = 4k, \\
p_{rr} + 2p_{zz} = 0, \\
p_{\varphi\varphi} = p_{zz}.
\end{gathered}
\tag{16}
$$

2) subdomain $b_1 < r < b_2$

$$
\begin{gathered}
2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{zz} = 4k, \\
\sigma_{rr} + \sigma_{\varphi\varphi} - 2\sigma_{zz} = 4k, \\
p_{rr} + p_{zz} + p_{\varphi\varphi} = 0.
\end{gathered}
\tag{17}
$$

3) subdomain $b_2 < r < b_3$

$$
\begin{gathered}
\sigma_{rr} + \sigma_{\varphi\varphi} - 2\sigma_{zz} = 4k, \\
2p_{rr} + p_{zz} = 0, \\
p_{\varphi\varphi} = p_{rr}.
\end{gathered}
\tag{18}
$$

The plastic deformations are obtained by the equations (1), (4), (5), (16), (17), (18) for domain $R_1 < r < b_1$:

$$
\begin{aligned}
p_{rr}^{(1)} &= \frac{(1-4\chi^2)}{3\chi q}(G_{-\chi}(r,t) - G_\chi(r,t)) - \\
&- \frac{(2\chi+1)(\chi+1)}{2\chi}F_\chi(r,t) + \frac{(2\chi-1)}{3}r^{\chi-1}c_1(t) + \\
&+ \frac{(2\chi-1)(\chi-1)}{2\chi}F_{-\chi}(r,t) - \frac{(2\chi+1)c_2(t)}{3r^{\chi+1}} + \\
&+ 2\Delta(r,t) - \frac{2k(r,t)}{\omega}, \\
\omega &= \frac{\mu q}{(\lambda+2\mu)}, \quad \chi = \sqrt{1 + \frac{3\mu}{q}};
\end{aligned}
\tag{19}
$$

for domain $b_1 < r < b_2$:

$$
\begin{aligned}
p_{rr}^{(2)} &= \frac{2G_1(r,t)}{q} - \frac{2G_{-1}(r,t)}{3q} + \frac{c_3(t)}{2q} - \\
&- 3F_1(r,t) - \frac{2k(r,t)}{\omega} + 2\Delta(r,t) + \frac{c_4(t)}{r^2},
\end{aligned}
\tag{20}
$$

$$
p_{zz}^{(2)} = \frac{4}{3q}G_{-1}(r,t) - \frac{c_3(t)}{q} + \frac{4k(r,t)}{3q} - \Delta(r,t);
$$

for domain $b_2 < r < b_3$:

$$
p_{rr}^{(3)} = \frac{\Delta(r,t)}{2\chi^2} - \frac{(\lambda+2\mu)k(r,t)}{\mu(q+3\mu)} + c_5(t).
\tag{21}
$$

The unknown functions $c_i(t)$ presented in the eqs. (19), (20), (21) are found from the boundary conditions (3) and continuity conditions of the deformations (1) at the elastic-plastic borders b_i. The position of the elastic-plastic borders is computed by the system of equations:

$$
\begin{gathered}
2p_{rr}^{(2)}(b_2,t) + p_{zz}^{(2)}(b_2,t) = 0, \\
p_{rr}^{(2)}(b_1,t) + 2p_{zz}^{(2)}(b_1,t) = 0, \\
p_{rr}^{(3)}(b_3,t) = 0.
\end{gathered}
\tag{22}
$$

IV. Thermal and Residual Stresses Computations

Thermal stresses can be presented under given temperature as the functions of the plastic deformations p_{rr}, $p_{\varphi\varphi}$. These functions are the same for any yield criterion and are furnished as

$$
\begin{aligned}
\sigma_{rr} &= \frac{2\mu}{\eta^2}\int_{R_1}^{r} \frac{p_{rr}(\rho,t) - p_{\varphi\varphi}(\rho,t)}{\rho}d\rho + c_6(t) - \\
&- \frac{2\mu^2}{(\lambda+2\mu)r^2}\int_{R_1}^{r}\rho(p_{rr}(\rho,t) + p_{\varphi\varphi}(\rho,t))d\rho + \\
&+ \frac{c_7(t)}{r^2} - 2\omega F_1(r,t), \quad \sigma_{\varphi\varphi} = (r\sigma_{rr}(r,t)_{,r}), \\
\sigma_{zz} &= \omega(p_{rr}(r,t) + p_{\varphi\varphi}(r,t)) + \\
&+ \frac{\lambda\sigma_{rr}(r,t) + \lambda\sigma_{\varphi\varphi}(r,t) - 2\mu\Delta(r,t)}{2(\lambda+\mu)}.
\end{aligned}
\tag{23}
$$

The thermal stresses fields for each piecewise linear yield criterion under given thermal action are shown on Fig. 2. We

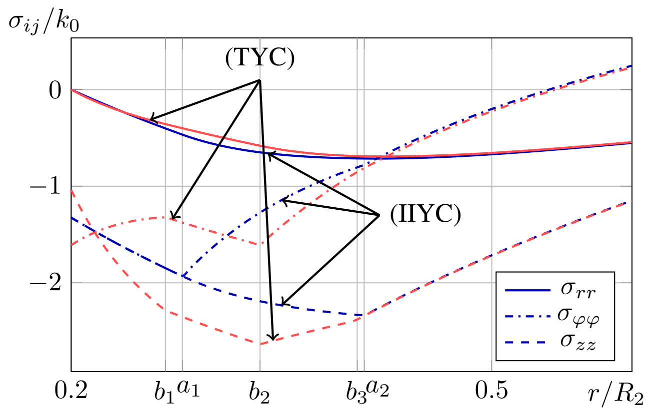

Fig. 2. The thermal stresses corresponding to the Tresca (TYC) and the Ishlinsky–Ivlev (IIYC) piecewise yield criteria. $R_1/R_2 = 0.2$

can simply obtain the equations of the residual stresses by vanishing of the terms with the thermal expansion function $\Delta(r,t)$ in the eqs. (23). The residual stresses are shown in the Fig. 3 for the Tresca and the Ishlinsky–Ivlev yield criteria under referential temperature. Arithmetic mean of the solutions for the Tresca and the Ishlinsky–Ivlev yield criteria can predict the strain-stress state calculating in the frameworks of the von Mises yield criterion. The thermal stress states in these cases are shown on the Fig. 4.

V. Conclusion

The obtained results can be generalized to the case of non-stationary temperature action. Numerical algorithms for calculating stresses within the framework of the von Mises

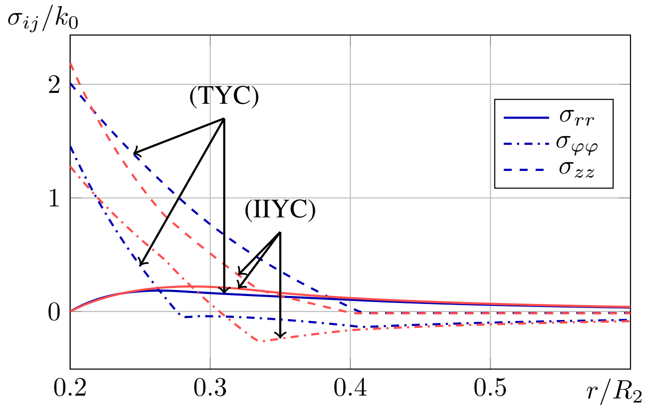

Fig. 3. The residual stresses fields corresponding to the residual deformations for the Tresca (TYC) and the Ishlinsky–Ivlev (IIYC) yield criteria. $R_1/R_2 = 0.2$

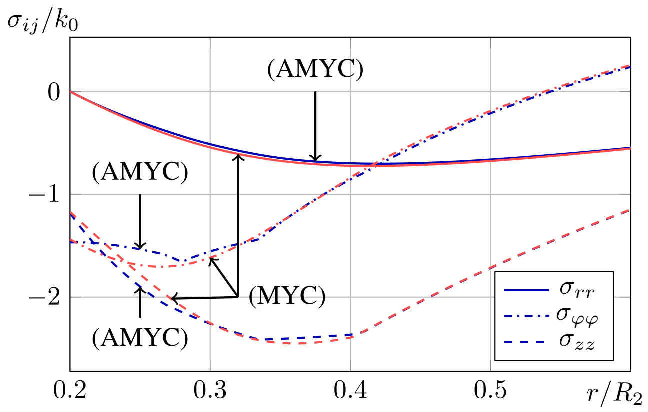

Fig. 4. Thermal stresses. Arithmetic mean (AMYC) of the solutions for the Tresca and the Ishlinsky–Ivlev yield criteria. The numerical solution (MYC) for the von Mises yield criterion. $R_1/R_2 = 0.2$

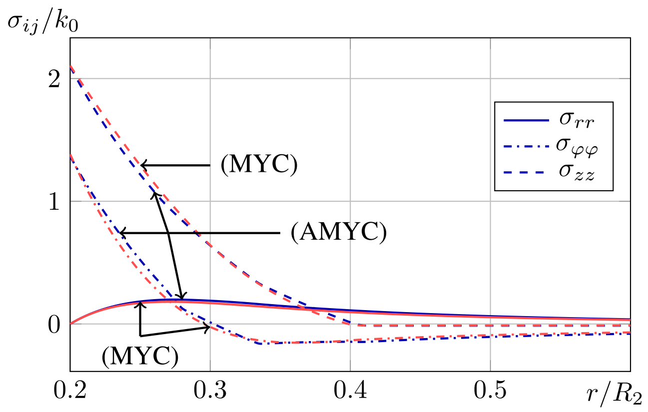

Fig. 5. Residual stresses. Arithmetic mean (AMYC) of the solutions for the Tresca and the Ishlinsky–Ivlev yield criteria. The numerical solution (IIYC) for the von Mises yield criterion. $R_1/R_2 = 0.2$

yield criterion taking into account the local time derivative of temperature require a large calculating time. The replacement of such numerical solutions by approximations being arithmetic mean (AMYC lines on the Figs 4, 5) of the solutions for the Tresca and the Ishlinsky–Ivlev yield criteria can simplify the problem of calculating of stress-strain state parameters. The alternative solutions make it possible to estimate the size of the plastic flow domain and the residual stresses values. In addition, such solutions can be used to evaluate the correctness of the numerical solutions obtaining under the von Mises yield criterion.

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