

PAPER • OPEN ACCESS

Numerical Analysis of the Elastic-Plastic Boundaries in the Thermal Stresses Theory Frameworks

To cite this article: E V Murashkin *et al* 2017 *J. Phys.: Conf. Ser.* **937** 012030

View the [article online](#) for updates and enhancements.

Related content

- [Simulation of X-Ray Mask Distortion](#)
Masatoshi Oda, Shigehisa Ohki, Akira Ozawa et al.
- [A photoelastic determination of the elastic-plastic boundary](#)
E E Gdoutos
- [Strain-stress simulation and comparison of different welding sequences during manufacturing of packing vacuum cover](#)
Yanhu Wang, Xizhang Chen and Sergey Kononov

Numerical Analysis of the Elastic-Plastic Boundaries in the Thermal Stresses Theory Frameworks

E V Murashkin^{1,2,3}, E P Dats⁴, V V Klindukhov¹

¹ National Research Nuclear University “MEPhI” (Moscow Engineering Physics Institute), Kashirskoye shosse 31, Moscow, 115409, Russia

² Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Vernadsky Ave 101 Bldg 1, Moscow, 119526, Russia

³Bauman Moscow State Technical University, 2nd Baumanskaya Str 5/1, Moscow, 105005, Russia

⁴ Vladivostok State University of Economics and Service, Gogolya str. 41, Vladivostok, 690014, Russia

E-mail: evmurashkin@gmail.com

Abstract. The present study is devoted to the problem of the thermal stresses theory for a hollow thermo-elastic-plastic ball under unsteady heating. Throughout the paper the conventional elastic-plastic Prandtl-Reuss model generalized on the thermal effects is used with the Tresca yield criterion linearly depending on temperature. A numerical-analytical algorithm for calculating the elastic-plastic boundaries positions and irreversible deformations is proposed. The strain-stress states during the plastic flow, repeated plastic flow and unloading are analyzed. The strain-stress state parameters like residual stresses and displacements are computed and analyzed.

1. Introduction

The prediction of residual stresses caused by uneven heating of elastoplastic material is one of the most important modern engineering problem. The main approach here is the solutions taking into account the plastic properties of materials under thermal action with the assumption of the independence of the temperature field from deformations [1-17]. Some results of the similar problems were used in the studies concerning residual stress calculations within the surface growth theory frameworks in additive manufacturing technologies (e.g., see [18, 19]) and geomechanics (e.g., see [20]). A number of problems were investigated for a stationary distributed temperature field. These solutions were constructed by comparison with isothermal elastoplastic problems solutions. New qualitative solutions can be obtained within the frameworks of the temperature stresses theory when the stresses occurs in virtue of a non-stationary thermal field gradient. The deformed state of the material can simultaneously satisfy to conditions of plastic flow and unloading in different domains. Elastoplastic boundaries separating such domains of irreversible deformation appear and disappear, moving in the direction of the thermal gradient increasing. The dependence of the yield stress on temperature leads to the appearance of high levels of residual deformations that can cause the reverse plastic flow during unloading. The study of these problems allow us to more accurately predict the distribution of residual stresses and deformations during high temperature metal forming. The processes of arising, development and disappearance of plastic flows during heating cause the necessity of elastoplastic boundaries localisation and irreversible (residual) deformations computing within unloading domains. The problems when the elastoplastic boundaries depend on the deformation process



and are completely determined by the known temperature field are discussed in depth in [3-6,13,14]. They consider the problems of heating symmetric elastoplastic solids [3-6] and plates [2]. In the present paper the residual deformation within the unloading domain is derived by an envelope of the possible plastic deformations family within the plastic flow domain undependent on the residual deformations. The solution of the problem of non-stationary heating of hollow elastoplastic solids is more complicated. In this case, the boundaries position depends on the level of residual deformations and must be determined due to a system of integro-differential equations defining the conditions for the stress-strain state parameters continuity. In this paper, a numerical-analytical technique for determining the elastoplastic boundaries spatial localisation in a one-dimensional problem of heating a hollow elastoplastic ball is proposed. It allows one to calculate the positions of the elastoplastic boundaries and the distribution of plastic deformations under conditions of a linear dependence of the yield stress on temperature with high accuracy and speed. The case reverse plastic flow in the heating domain is also considered. Distributions of residual stresses and displacements forming during cooling of the material are computed.

2. Governing Equations. Boundary Value Problem Statment

Hereafter let use the conventional Prandtl–Reuss elastic-plastic model generalized on thermal effects [1, 2]. Consider thermoelastoplastic hollow ball with inner and outer radia R_0 and R_1 respectively under referential temperature $T = T_0$ and under thermal expansion conditions

$$\sigma_{rr}(R_0) = 0, \quad \sigma_{rr}(R_1) = 0, \quad T_{,r}(R_0) = 0, \quad T(R_1) = T_0(1 - e^{-xt}). \quad (1)$$

Total deformations can formulae depending on the radial displacement

$$e_{rr} = u_{r,r}, \quad e_{\phi\phi} = e_{\theta\theta} = \frac{u_r}{r}. \quad (2)$$

Then according to eqs. (2) we obtain the relationship between components of the total strain tensor

$$e_{rr} = u_{r,r} = \left(\frac{ru_r}{r} \right)_{,r} = (re_{\phi\phi})_{,r}. \quad (3)$$

The equilibrium equation in the spherical symmetry case is read by

$$\sigma_{rr,r} + \frac{2(\sigma_{rr} - \sigma_{\phi\phi})}{r} = 0, \quad (4)$$

and the components of stress tensor are coupled by formulae

$$\sigma_{\phi\phi} = \sigma_{\theta\theta} = \frac{1}{2r} (r^2 \sigma_{rr})_{,r}. \quad (5)$$

The constitutive strain–stress equations in the spherical symmetry case are formulated by

$$e_{rr} = \alpha(T - T_0) + \frac{(\lambda + \mu)\sigma_{rr} - \lambda\sigma_{\phi\phi}}{\mu(3\lambda + 2\mu)}, \quad e_{\phi\phi} = \alpha(T - T_0) + \frac{(\lambda + 2\mu)\sigma_{\phi\phi} - \lambda\sigma_{rr}}{2\mu(3\lambda + 2\mu)}. \quad (6)$$

Throughout the present study we chose the Tresca yield criterion (maximum tangential stress one)

$$f = \max \{ |\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1| \} - 2k(T) = 0, \quad (7)$$

and the the yield stress k linearly dependent on temperature [2]

$$k = k_0 \left(1 - \frac{T - T_0}{T_m - T_0} \right), \quad (8)$$

wherein k_0 is the yield stress under referential temperature T_0 , T_m denotes the melting temperature of material.

3. Plastic Flow and Unloading Problems Solutions

The stress state corresponds to the edge of the Tresca's prism (7) under certain outer temperature. Then the size of plastic flow domain are given by inequalities $a(t) < r < R_1$, where $a(t)$ is the elastic-plastic boundary. Plastic deformations in this domain are furnished by

$$\begin{aligned} p_{rr} &= -2p_{\phi\phi} = 2\alpha T - \frac{8\alpha k}{\nu} - \frac{6\alpha}{r^3} \int_a^r T\rho^2 d\rho - \frac{2D(t)}{r^3}, \\ D &= \frac{3\alpha a^3}{(a^3 - R_0^3)} \left(\int_{R_0}^a T\rho^2 d\rho + \frac{4R_0^3}{\nu} \int_a^{R_1} \frac{k}{\rho} d\rho \right). \end{aligned} \quad (9)$$

The slowdown of plastic deformations increasing (9) is coupled with the levelling of the temperature gradient. At some point on the surface $r = R_1$ the following condition $\dot{p}_{rr} = 0$ will fulfilled meaning the termination of the yield criterion. Thus the unloading domain $b(t) < r < R_1$ is enlarged, where $b(t)$ denotes the unloading boundary. Then for the time dependent function $D(t)$ (9) the following equation can be obtained

$$D = J \left(\frac{2\alpha}{R_0^3} \int_{R_0}^a T\rho^2 d\rho + \frac{2\alpha}{R_1^3} \int_b^{R_1} T\rho^2 d\rho - \frac{2\alpha (R_1^3 - b^3)}{R_1^3 b^3} \int_a^b T\rho^2 d\rho + \frac{2\alpha}{\nu} \int_a^b \frac{k}{\rho} d\rho - \int_b^{R_1} \frac{p'_{rr}}{\rho} d\rho \right), \quad (10)$$

$$\text{where } J(a, b) = \frac{3a^3 b^3 R_1^3 R_0^3}{2(a^3 b^3 R_1^3 - R_0^3(a^3 b^3 + R_1^3(b^3 - a^3)))}.$$

The following nonlinear equations system can be solved to obtain the elastic-plastic boundaries positions and the function of irreversible deformation p'_{rr}

$$\begin{cases} p_{rr}(a, t) = 0, \\ p'_{rr}(b) = p_{rr}(b, t), \\ \dot{p}_{rr}(b, t) = 0. \end{cases} \quad (11)$$

Then one can approximate the integrals of deformation function p'_{rr} . Assume that the function of irreversible deformation $p'_{rr}(r)$ is smooth inside unloading domain. Then for a small time interval $\Delta t = t_1 - t_u$ the unloading domain is enlarged according to $\Delta b = b_1 - R_1$. The integrand $\frac{p'_{rr}(r)}{r}$ is similar to straight line in the interval Δb . This straight line passes through 2 points with coordinates $\left(b_1, \frac{p'_{rr}(b_1)}{b_1}\right)$ and $\left(R_1, \frac{p'_{rr}(R_1, t_u)}{R_1}\right)$. Integral $\int_{b_1}^{R_1} \frac{p'_{rr}}{\rho} d\rho$ is the case can be transformed by

$$\int_{b_1}^{R_1} \frac{p'_{rr}}{\rho} d\rho = I(b_1) = \frac{1}{2} (b_1 - R_1) \left(\frac{p'_{rr}(b_1)}{b_1} + \frac{p_{rr}(R_1, t_u)}{R_1} \right).$$

Substituting this expression into equations (11), and using the method of successive approximations, we find the values $p'_{rr}(b_1)$, $b(\Delta t + t_u) = b_1, a(\Delta t + t_u) = a_1$. At the i -th time step $i\Delta t + t_u$ integral can be rewritten as

$$\int_{b_i}^{R_1} \frac{p'_{rr}}{\rho} d\rho = I(b_i) = \frac{1}{2} (b_i - b_{i-1}) \left(\frac{p'_{rr}(b_i)}{b_i} + \frac{p'_{rr}(b_{i-1})}{b_{i-1}} \right) + I(b_{i-1}). \quad (12)$$

Consistently calculating the values a_i , b_i , $p'_{rr}(b_i)$, at each time step, we find the elastoplastic boundaries positions and the irreversible deformation inside the unloading area. The condition $a(t_k) = B(t_k)$ is fulfilled during the calculation at some time meaning the ending of the irreversible deformation process.

Assume that at the time $t = t_r > t_u$ on the inner surface the yield criterion (7) is valid. Then for a time $t > t_r$ within vicinity of the inner surface ($R_1 < r < c(t)$) the plastic flow domain is developed, where $c(t)$ is the elastoplastic boundary. Thus, there are both plastic flow and thermoelastic deformation (unloading) domains. The strain-stress state parameters within domain ($c(t) < r < R_2$) are derived by equations

$$\begin{aligned} \sigma_{rr} &= 4 \int_c^r \frac{k(\rho, t)}{\rho} d\rho + G(t), & \sigma_{\varphi\varphi} &= 4 \int_c^r \frac{k(\rho, t)}{\rho} d\rho + 2k(r, t) + G(t), \\ u_r &= \frac{4r}{(3\lambda + 2\mu)} \int_c^r \frac{k(\rho, t)}{\rho} d\rho + \frac{3}{r^2} \int_{R_1}^r \Delta(\rho, t) \rho^2 d\rho + \frac{G(t)r}{(3\lambda + 2\mu)} + \frac{H(t)}{r^2}, \\ p_{rr} &= 2\Delta(r, t) - \frac{6}{r^3} \int_{R_1}^r \Delta(\rho, t) \rho^2 d\rho + \frac{2k(r, t)}{\omega} - \frac{2H(t)}{r^3}. \end{aligned} \quad (13)$$

For time dependent functions taking into account the new dependencies (13) we have following relations

$$\begin{aligned} m &= \frac{3a^3b^3c^3R_1^3}{R_1^3(b^3c^3 - a^3(c^3 - b^3)) - a^3b^3c^3}, \\ W(t) &= m \left(\frac{1}{c^3} \int_{R_1}^c \Delta(\rho, t) \rho^2 d\rho + \frac{1}{a^3} \int_{R_1}^a \Delta(\rho, t) \rho^2 d\rho - \right. \\ &\quad \left. - \frac{1}{b^3} \int_{R_1}^b \Delta(\rho, t) \rho^2 d\rho + \frac{1}{\omega} \int_a^b \frac{k(\rho, t)}{\rho} d\rho - \frac{1}{\omega} \int_c^{R_2} \frac{k(\rho, t)}{\rho} d\rho \right), \\ D(t) &= \frac{m}{2} \int_b^{R_2} \frac{p'_{rr}(\rho)}{\rho} d\rho - W(t), \\ A &= \frac{4\omega}{3c^3} D(t) + 4 \int_{R_1}^c \frac{k(\rho, t)}{\rho} d\rho + \frac{4\omega}{3c^3} \int_2^{R_1} \frac{p'_{rr}(\rho)}{\rho} d\rho c, \\ B &= -\frac{4}{3}\omega D(t), \quad F = -\frac{4}{3}\omega D(t), \quad H = D(t), \\ E &= -2\omega \int_b^{R_2} \frac{p'_{rr}(\rho)}{\rho} d\rho + \frac{4\omega}{3R_2^3} \int_{R_1}^{R_2} \Delta(\rho, t) \rho^2 d\rho + \frac{4\omega D(t)}{3R_2^3}, \\ C &= 4 \int_{R_1}^c \frac{k(\rho, t)}{\rho} d\rho - \frac{4\omega}{a^3} \int_{R_1}^a \Delta(\rho, t) \rho^2 d\rho - \\ &\quad - \frac{4\omega(c^3 - a^3)D(t)}{3a^3c^3} + \frac{4\omega}{c^3} \int_{R_1}^c \Delta(\rho, t) \rho^2 d\rho. \end{aligned} \quad (14)$$

The localisations of elastoplastic boundaries $a(t)$, $b(t)$, $c(t)$ and irreversible deformation $p'_{rr}(r)$ can be obtained by numerically solution of the system of equations (11) and adding equation which determines the vanishing of plastic deformation

$$p_{rr}(c, t) = 0. \quad (15)$$

Initial approximations for the system (11), (15) are given by the values $c_0 = R_2$, $a_0 = a(t_r)$, $b_0 = b(t_r)$, $p'_{rr}(b_0) = p'_{rr}(b(t_r))$.

Fig. 1 shows the stresses at the moment of the disappearance of the first plastic flow and the development of a repeated plastic flow inside the outer surface vicinity

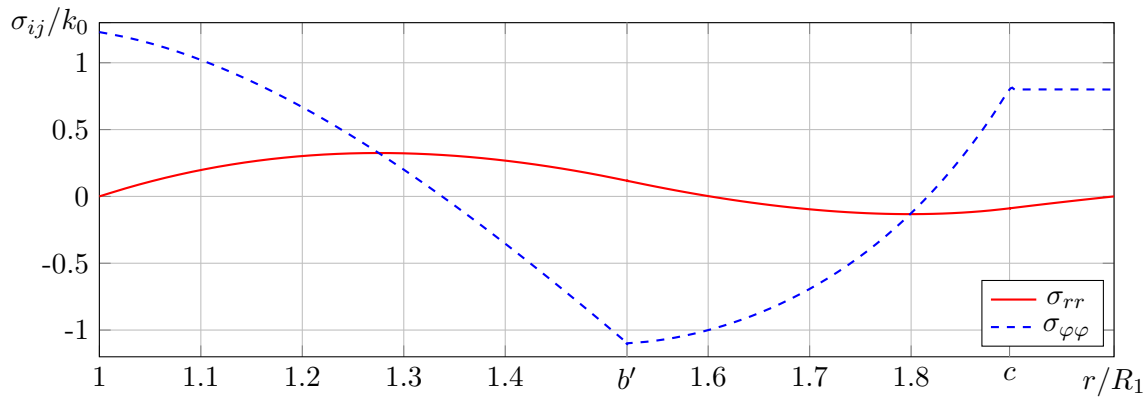


Figure 1. Field of stresses in a continuous sphere in case of occurrence of repeated plastic flow on the external surface, $R_2/R_1 = 2$, b' is the final boundary position of the unloading domain

The strain-stress state within the domain of the repeated plastic flow ($c(t) < r < R_2$) transforms into a state of neutral loading under gradual equalization of the temperature field. The repeated plastic flow boundary occupies the limiting position $c(t) = c'$. The subsequent unloading of the material occurs as a result of an increasing of the yield strength under uniform cooling. Stresses of the fully heated ball under the given temperature ($\Delta_{,r}(r, t) = 0$) are the residual ones (Fig. 2).

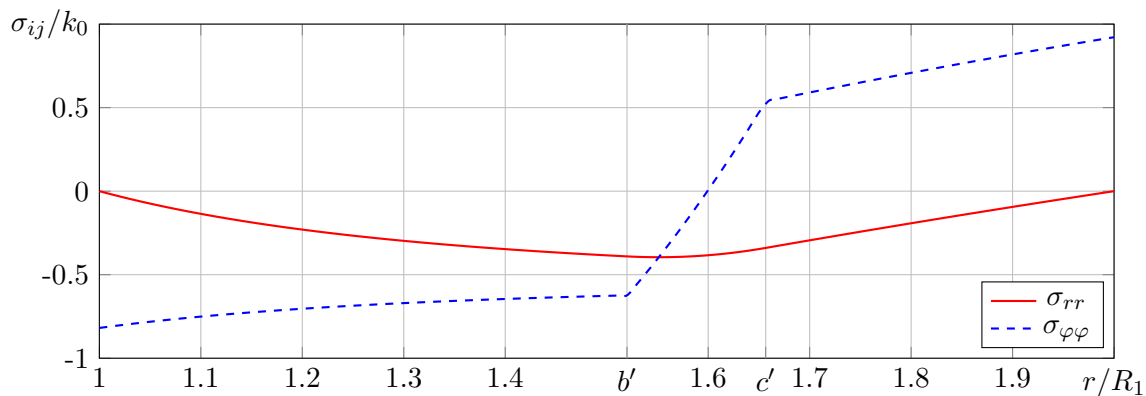


Figure 2. Residual stresses at zero temperature gradient (maximum temperature / referential temperature)

Fig. 3 shows us the radial distribution of the displacement after a uniform cooling of the ball to the referential temperature ($\Delta(r, t) = 0$)

From Fig. 3 it follows that in the case of the development of irreversible deformation processes in the material of a hollow sphere, its size after cooling to the initial temperature will be smaller than the referential size.

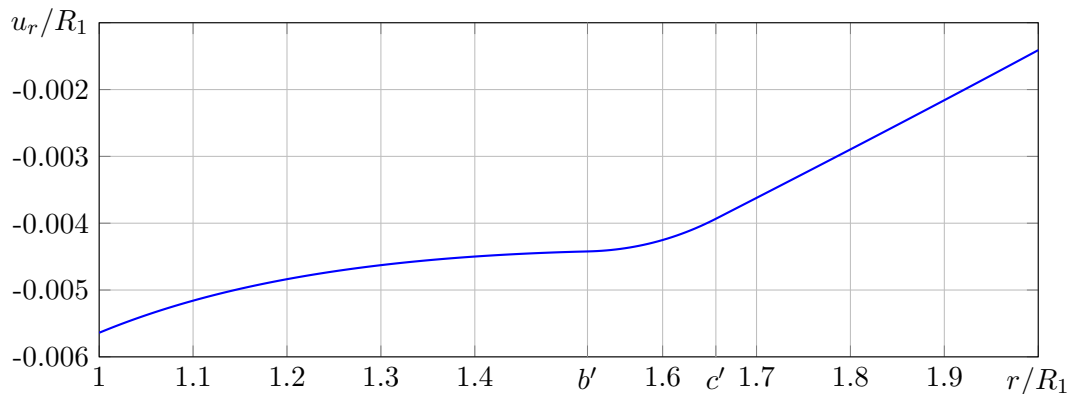


Figure 3. Residual radial displacement at referential temperature

Acknowledgment

The research was financially supported by RSF (project No. 17-19-01257).

Conclusion

- The boundary value problem within frameworks of the the conventional Prandtl–Reuss elastic-plastic model generalized on thermal effects have been solved.
- The hollow elastic-plastic ball under central symmetry unsteady heating have been considered. The numerical-analytical algorithm for calculating the elastic-plastic boundaries positions and irreversible deformations has been proposed.
- It has been shown that the strain-stress state within the domain of the repeated plastic flow has been transformed into a state of neutral loading during the temperature field equalization.
- It has been shown that the accumulated irreversible deformation in a hollow sphere lead to decreasing its size after cooling to the referential temperature.

References

- [1] Boley B and Weiner J 1960 *Theory of Thermal Stresses* (New York: Wiley)
- [2] Burenin A, Dats E and Murashkin E 2014 *Mech. of Solids* **49** 218
- [3] Dats E, Mokrin S and Murashkin E 2015 *Lect. Not. Eng. and Comp. Sc.* **2218** 1043
- [4] Dats E, Mokrin S and Murashkin E 2016 *Key Eng. Mat.* **685** 37
- [5] Dats E and Murashkin E 2016 *Lect. Not. Eng. and Comp. Sc.* **2223** 69
- [6] Dats E, Murashkin E and Stadnik N 2017 *Proc. Eng.* **173** 891
- [7] Gamer U 1967 *Arch. Appl. Mech.* **36(3)** 174
- [8] Gamer U 1988 *J. Therm. Str.* **13** 159
- [9] Guven U and Altay O 2000 *Int. J. Mech. Sc.* **42** 831
- [10] Odeno H 1969 *Acta Polytech. Scand. Phys. Incl. Nucleon. Series* **66** 243
- [11] Odeno H 1969 *J. Mech. Eng. Sc.* **2** 284
- [12] Orcan Y 1991 *Acta Mech.* **90** 61
- [13] Orcan Y 1995 *Int. J. Mech. Sc.* **33** 1689
- [14] Parkus H 1954 *Proc. Sec. U.S. Nat. Congr. Appl. Mech.* **32** 307
- [15] Parkus H 1959 *Arch. Appl. Mech.* **28(1)** 251
- [16] Perzyna P and Sawczuk A 1973 *Nucl. Eng. and Design* **24** 1-55
- [17] Raniecki B 1966 *Rozpr. Inz.* **14(3)** 479
- [18] Manzhirov A V 2013 *Key Eng. Mat.* **535** 89
- [19] Manzhirov A V and Parshin D A 2015 *Mech. Solids* **50(6)** 657
- [20] Manzhirov A V and Parshin D A 2016 *Lec. Not. Eng. and Comp. Sc.* **2224(1)** 1131