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Nonlinear Analysis of Outward Propagating Hydrodynamically Unstable Flame at Large Gas Expansion Ratio

S. S. Minaev[a,b,*], V. V. Gubernov[a], and E. P. Dats[c]

[a] *Lebedev Physical Institute, Russian Academy of Sciences, Mocow, 119991 Russia*

[b] *Far Eastern Federal University, Vladivostok, 690922 Russia*

[c] *Vladivostok State University, Vladivostok, 690014 Russia*

**e-mail: minaevss@yahoo.com*

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Abstract—In the context of the large thermal-expansion approximation, we derive an equation describing outward propagating flame under conditions of Darrieus–Landau instability. We show that the second-order theory leads to system of two evolution equations for the flame front perturbations and for the potential of the unburned mixture flow. In the limiting case of long evolution, the system of equations can be reduced to one equation similar to the Sivashinsky equation.

Keywords: hydrodynamic instability, expanding flame, evolutionary equation, large gas expansion coefficient

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INTRODUCTION

The dynamics of a premixed flame propagation in a large volume of a combustible mixture is one of the interesting fundamental problems of combustion theory, which is associated with many important applications, from gas combustion in internal combustion engines to large-scale explosions of gas mixtures. The growth of the flame total surface due to hydrodynamic instability increases the speed of flame propagation and determines the dynamics of energy release, changes in pressure and other characteristics in the system with gas combustion. Therefore, modeling of flame front dynamics is an urgent problem for practical applications. Numerical simulations of a cellular flame dynamics and the associated disturbed gas flow require significant computational costs, which depend on the size of the computational domain. For example, simulations of a diverging cylindrical flame [1], show that the number of cells at the flame front is proportional to the flame radius. Therefore, the number of grid nodes required to calculate the gas flow field increases as a square of the flame radius. One possible way to reduce the amount of computation is to use approximate evolutionary equations for the variables given at the flame front, which avoids computations in the space surrounding the flame. In the linear approximation, the evolutionary equation describes an unlimited growth of the perturbations that follows from the classical Landau–Darrieus theory of linear hydrodynamic instability [2, 3]. To describe the nonlinear stabilization of the perturbation's growth and flame propagation velocity, it is necessary to take into account nonlinear terms [4]. The solution of this problem for arbitrary values of the gas expansion coefficient $E = \rho_1/\rho_2$, (where ρ_1 and ρ_2 are the densities of the combustible gas and the combustion products) encounters significant difficulties associated with the nonlinearity of the equations describing vortex flow in the combustion products. Assuming that the gas expansion coefficient is close to unity $(E-1) \ll 1$, one can obtain nonlinear integro-differential equation, known as the Sivashinsky equation [5], which is widely used not only in combustion theory, but also in other areas of physics. A remarkable property of this equation is that it admits a whole family of exact analytical solutions [6–8] and describes many complex nonlinear processes, for example, the self-acceleration of a diverging flame [1]. The assumption about the smallness of the expansion coefficient makes it problematic to obtain quantitative estimates of the propagation velocity and the other characteristics of cellular flame, since under normal conditions the typical gas expansion coefficients have values of $E \sim 5$–10. For mixtures of combustible gas with liquid droplets or during liquid combustion, the parameter E can reach essentially larger values due to liquid evaporation. In this paper, we consider case of large gas expansion coefficients and use an asymptotic expansion in which the quantity $1/E \ll 1$ is

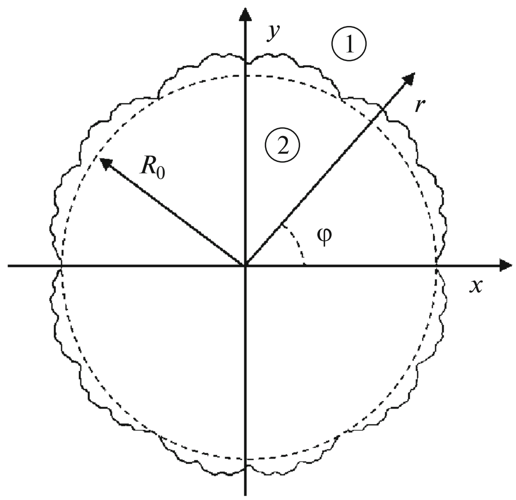

Fig. 1. Scheme of outward propagating cylindrical flame.

a small parameter. Note that in the general case of a finite value of the expansion coefficient, there is no solution to the problem. In the limiting case, when the expansion coefficient is close to unity, the solution to the problem was obtained in the work of Sivashinsky [5]. Gas combustion with such an expansion coefficient is not typical, however, the qualitative behavior of systems with a finite expansion coefficient turned out to be very close to that which follows from the analysis with an expansion coefficient close to unity. This fact served as motivation for considering the limiting case, when the expansion coefficient is not close to unity, but, on the contrary, is very large.

The paper considers the case of a diverging cylindrical flame, for which the flame front evolutionary equation is derived in the approximation of a large gas expansion coefficient.

MATHEMATICAL MODEL

The equations for the gas velocity $\mathbf{V} = (V, U)$, where V and U are, respectively, the radial and tangential velocity components in a cylindrical coordinate system (r, φ), read:

$$\operatorname{div}\mathbf{V} = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\mathbf{V} + \mathbf{V}\nabla\mathbf{V} = -\frac{1}{\rho}\nabla P. \tag{2}$$

Since the flame front separates two regions with different densities ρ, the equations of motion differ in the combustible mixture and the combustion products. In the following, indices 1 and 2 are used to mark, the combustible mixture and the combustion products, respectively. The flame front is determined by expression $r = R(r, \varphi, t) = R_0 + f(\varphi, t)$, where $R_0 = ES_0t$ the radius of expanding cylindrical flame. The $E = \rho_1/\rho_2 > 1$ is the ratio of gas densities or gas expansion coefficient and S_0 is the speed of plane laminar flame. The scheme of the flame is given in Fig. 1.

Boundary conditions at the flame front $r = R$ have the form:

$$D_n - V_{n1} = S_b, \tag{3}$$

$$E(D_n - V_{n1}) = D_n - V_{n2}, \tag{4}$$

$$V_{\tau 1} = V_{\tau 2}, \tag{5}$$

$$P_1 - P_2 = \rho_1(E-1)S_b^2. \tag{6}$$

Here D_n is the flame front velocity along the normal to the flame surface, which is towards combustible mixture. V_n and V_τ are the normal and tangential components of gas velocity, respectively. We use the phenomenological dependence of the flame propagation velocity S_b on the local curvature of the two-dimensional flame front $f_{\varphi\varphi}/R_0^2$:

$$S_b(F) = S_0(1 + \sigma f_{\varphi\varphi}/R_0^2). \tag{7}$$

Here σ is the Markstein length [9] and S_0 is the speed of plane flame. In the case $\sigma > 0$, such dependence leads to the stabilization of small-scale perturbations. The solutions of Eqs. (1), (2) with boundary conditions (3)–(6) for unperturbed cylindrical flame $r = R_0 = ES_0t$ read:

$$\begin{gathered} V_{10} = (E-1)S_0\frac{R_0}{r}, \quad V_{20} = U_{10} = U_{20} = 0, \\ P_{10} = P_0 - \rho_1(E-1)S_0^2\left(E\ln\left(\frac{r}{R_0}\right) + \frac{(E-1)R_0^2}{r^2}\right), \quad P_{20} = P_0 - \rho_1 S_0^2\frac{(E^2-1)}{2}. \end{gathered} \tag{8}$$

EQUATIONS FOR PERTURBED FLAME

The asymptotic method applied in paper [10] for the case of plane flame is used to derive the equation for the perturbed cylindrical flame at large gas expansion coefficients. Equations (1), (2) and the boundary conditions (3)–(6) are written for the dimensionless components of the gas velocities $v_{1,2}$, $u_{1,2}$ and pressures $p_{1,2}$:

$$\begin{gathered} v_1 = (V_1 - V_{10})\varepsilon/S_0, \quad u_1 = U_1\varepsilon/S_0, \quad v_2 = V_2\varepsilon^2/S_0, \quad u_2 = U_2\varepsilon^2/S_0, \\ p_1 = \varepsilon^2(P_1 - P_{10})/\rho_1 S_0^2, \quad p_2 = \varepsilon^2(P_2 - P_{20})/\rho_1 S_0^2. \end{gathered} \tag{9}$$

Here $\varepsilon = E^{-\frac{1}{2}}$ and ε is considered to be a small parameter $\varepsilon \ll 1$. The radius and time are dimensioned, correspondingly, by Markstein length σ and by characteristic time $\frac{\varepsilon\sigma}{S_0}$. The nondimensional radius is denoted as $\eta = r/\sigma$, and the nondimensional time is $\tau = \frac{tS_0}{\varepsilon\sigma}$. The flame front in nondimensional variables reads $\eta = \eta_0 + F(\varphi,\tau)$, where $\eta_0 = \frac{R_0}{\sigma}$. The velocities $v_{1,2}$, $u_{1,2}$ and pressures $p_{1,2}$ are related with the small amplitude flame front perturbations. In the case of fresh mixture, denoted by index 1, the variables $P = P_{10} + \rho_1 S_0^2 p_1$ and $\mathbf{V} = (V_{10} + v_1S_0/\varepsilon, u_1S_0/\varepsilon)$ are substituted in the Eqs. (1), (2) to obtain the equations for nondimensional perturbations. The equations for combustion products perturbations are obtained by substitutions $P = P_{20} + \rho_1 S_0^2 p_2$ and $\mathbf{V} = (v_2S_0/\varepsilon^2, u_2S_0/\varepsilon^2)$ into Eqs. (1), (2). Then, the all terms having smallness ε in comparison with others are discarded in resulting equations.

The flow in the fresh mixture is determined by potential Φ, satisfying the Laplace equation, and the velocity components have the form:

$$v_1 = \Phi_\eta, \quad u_1 = \frac{1}{\eta}\Phi_\varphi, \quad \text{where} \quad \Phi_{\eta\eta} + \frac{1}{\eta}\Phi_\eta + \frac{1}{\eta^2}\Phi_{\varphi\varphi} = 0. \tag{10}$$

In the case of a cylindrical flame $\eta = \eta_0$ the radial velocity v_1 can be obtained from solution of the Laplace equation for the potential Φ:

$v_1 = \left.\frac{\partial\Phi}{\partial\eta}\right|_{\eta=\eta_0} = -\frac{\hat{K}\{\psi\}}{\eta_0}$, where $\hat{K}\{f\} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}|m|\int_{-\pi}^{\pi} f(\varphi^*)e^{\Im(\varphi-\varphi^*)}d\varphi^*$ is linear integral operator and $\psi(\varphi,\tau)$ is the value of the potential Φ at the cylindrical flame front. The calculations of the velocities and pressure approximate values at the perturbed flame front can be obtained by method proposed in paper [10]. The relation of radial velocity and the potential at the perturbed flame front follows from boundary condition (3) and it can be written as:

$$v_1 = \left.\frac{\partial\Phi}{\partial\eta}\right|_{\eta=\eta_0+F} = -\frac{\hat{K}\{\psi\}}{\eta_0} - \frac{\hat{K}\{F\hat{K}\{\psi\}\}}{\eta_0^2} - \frac{1}{\eta_0^2}\psi_{\varphi\varphi}F. \tag{11}$$

In the same way, one can calculate the tangential velocity u_1 and using Bernoulli's law, the pressure p_1:

$$u_1 = \frac{1}{\eta_0}\left(\psi_\varphi + \frac{\hat{K}\{\psi\}}{\eta_0}F_\varphi\right), \quad p_1 = -\psi_\tau - \frac{\hat{K}\{\psi\}F_\tau}{\eta_0} - \frac{1}{2}\left(\left(\frac{\hat{K}\{\psi\}}{\eta_0}\right)^2 + \frac{1}{\eta_0^2}\psi_\varphi^2\right). \tag{12}$$

In the approximation of large gas expansion coefficients, the equations for combustion products become quasi-stationary and do not contain time derivatives. This approximation essentially simplifies the solution of the problem. In particularly, the pressure can be divided into two components $p_2 = p_p + p_v(\chi)$, one of which is similar to pressure in potential flow $p_p = -v_2^2/2 - u_2^2/2$, and the other component p_v appears due to vortex flow and it depends only on the variable $\chi = \int_\eta^F u_2(\varphi, \eta_1)d\eta_1$.

Substituting fresh gas velocity v_1 in the form (11) into the boundary condition (3), one can obtain the first equation relating the function ψ with the flame front disturbances F:

$$F_\tau = -\frac{\hat{K}\{\psi\}}{\eta_0} - \frac{\hat{K}\{F\hat{K}\{\psi\}\}}{\eta_0^2} - \frac{F\psi_{\varphi\varphi}}{\eta_0^2} - \frac{F_\varphi\psi_\varphi}{\eta_0^2}. \tag{13}$$

The second equation for ψ and F can be obtained by the same method as used in [10] to derive the equation for perturbed plane flame:

$$\psi_\tau = \frac{-\hat{K}\{\psi\}}{\eta_0} - \frac{2F_{\varphi\varphi}}{\eta_0^2} - \frac{(\hat{K}\{F\})^2}{2\eta_0^2} - \frac{F_\varphi^2}{2\eta_0^2} + \frac{\hat{K}\{F\hat{K}\{F\}\}}{\eta_0^2} + \frac{FF_{\varphi\varphi}}{\eta_0^2} - \frac{\hat{K}\{\psi\}F_\tau}{\eta_0} - \frac{(\hat{K}\{\psi\})^2}{2\eta_0^2} - \frac{\psi_\varphi^2}{2\eta_0^2}. \tag{14}$$

Introducing new variables $X = F - \psi$ and $Y = F + \psi$, one can get that the variable X exponentially increases whereas Y exponentially decreases and tends to zero in the course of long-time evolution. The condition $Y = F + \psi = 0$ means that $F = -\psi$, and in this case equation for flame front perturbation takes the form similar to the Sivashinsky equation:

$$F_\tau = \frac{\hat{K}\{F\}}{\eta_0} + \frac{F_{\varphi\varphi}}{\eta_0^2} + \frac{F_\varphi^2}{\eta_0^2}. \tag{15}$$

Note that the physical interpretations of the solutions in the limiting cases $E - 1 \ll 1$ and $E \gg 1$ are different. When the gas expansion coefficient is small, the nonlinear equation describes the evolution of flame front perturbations F propagating in potential flow both in the combustible mixture and in the combustion products. In the case of large gas expansion coefficients, the nonlinear equation allows describing the vortex flow in the combustion products. The Eq. (15) in dimension variables read:

$$R_t = S_0\left(E + \sqrt{E}\left(\frac{\hat{K}\{R\}}{R_0} + \frac{\sigma R_{\varphi\varphi}}{R_0^2} + \frac{R_\varphi^2}{R_0^2}\right)\right). \tag{16}$$

The results of the numerical solution of an equation of the same type as Eq. (16) can be found in [11].

CONCLUSIONS

A nonlinear equation describing flame front dynamics was derived in approximation of large thermal-expansion coefficient. We show that the second-order theory leads to system of two evolution equations for the flame front perturbations and for the potential of the unburned mixture flow. In the limiting case of long evolution, the system of equations can be reduced to one equation similar to Sivashinsky equation obtained early for the case of small gas expansion coefficients. It allows transferring all the results, obtained earlier for small gas expansion coefficients, to the case of large gas expansion coefficients with appropriate normalization of the problem parameters. The fact that the resulting nonlinear equations for the flame front in both limiting cases are identical, up to normalization, suggest hypothesis about identical qualitative dynamic behavior of the flame in the process of long evolution, independent of the gas expansion coefficient value. This allows, for example, transferring the results of studies of hydrodynamic instability [12, 13], obtained using the Sivashinsky equation, to the case of large gas expansion coefficients.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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