===== **MAGNETISM** =====

Bethe Approximation in the Ising Model with Mobile Impurities

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Abstract—The Bethe approximation as applied to a system consisting of magnetic and nonmagnetic atoms in the thermodynamic equilibrium has been considered. In this approximation, the dependences of the magnetization and Curie temperature on the concentration of magnetic atoms for the Ising model with mobile nonmagnetic impurities have been constructed and the limiting concentrations of the appearance of spontaneous magnetization in the ground state have been obtained. It has been established that the considered approximation for a one-dimensional chain is the exact solution.

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1. INTRODUCTION

Despite the fact that the investigations of diluted and disordered magnets have been actively performed [1−4], most authors mainly concentrate their attention on studying the influence of "frozen-in" impurities, which are arranged in lattice sites randomly and without correlation. However, when, for example, the composition of a magnet changes in the course of the chemical reaction [1], this means that magnetic atoms (or impurity atoms) can be displaced and, if the reaction occurs sufficiently slowly, the system will be in a state close to the thermodynamic equilibrium. In addition, when the temperature changes, nonmagnetic impurities should be redistributed over lattice sites, which can change magnetic properties.

In this paper, we consider the Bethe approximation as applied to the diluted Ising magnet with mobile atoms. The Bethe approximation for a pure magnet can be obtained as the solution of the Ising problem for the Bethe lattice (tree) [5] or as the relation between the magnetization of the central atom and the magnetization of the atom of the first coordination sphere [6]. However, we can also consider the Bethe approximation as one of self-consistent methods for which the general construction scheme was presented in our previous papers [7−9]. These methods are based on the averaging over local exchange fields. The procedure of the averaging over local fields can also be used to analyze the behavior of an alloy of two types of magnetic atoms [10], as well as to analyze systems in which the exchange integral is a continuous function of the interatomic distance [11, 12]. The authors of [13] already applied one variant of this procedure to analyze equilibrium states of the alloy of magnetic and nonmagnetic atoms in the absence of the external magnetic field. In present paper, we discuss another approximation for the same model, which can be considered as the generalization of the Bethe method. In

addition, it is shown that this approximation for the one-dimensional lattice is an exact solution similarly to the usual Bethe approximation for the case of a pure magnet.

2. ISING MODEL WITH MOBILE IMPURITIES AND THE APPROXIMATED METHODS OF ITS SOLUTION

Let us consider a crystal lattice with coordination number q, the sites of which can contain magnetic and nonmagnetic atoms (atoms of types 1 and 2, respectively). For each magnetic atom, there is the Ising spin $s_i = \pm 1$ so that the exchange interaction energy of two magnetic atoms with spins s_i and s_j is $-Js_is_j$ if the atoms are arranged in neighboring lattice sites and equals zero otherwise.

As was done in the study of binary alloys [5], we assume that, in the system, there are interatomic forces with the interaction potential of the type of the Lennard-Jones potential, which rapidly decreases at large distances [5]. Therefore, we will consider that their interaction radius is restricted by the first coordination sphere; we will further call these forces as the "Coulomb" forces. Let us denote the potential of these forces as $-U_{\alpha\beta}$ (α, $\beta = 1$, 2). Now, we assume that, for each lattice site, there is the variable σ_i which is equal to s_i, when this site is occupied by a magnetic atom, and is equal to zero, when it is occupied by a nonmagnetic atom. Then, exchange interaction energy E_{ex} and the Coulomb energy E_k can be written in the form of sums over all ordered pairs of neighboring sites

$$E_{ex} = -\sum_{(i,j)} J\sigma_i\sigma_j,$$

$$E_k = -\sum_{(i,j)} \{ U_{11}\sigma_i^2\sigma_j^2 + U_{22}(1-\sigma_i^2)(1-\sigma_j^2)$$
$$+ U_{12}[\sigma_i^2(1-\sigma_j^2) + \sigma_j^2(1-\sigma_i^2)]\}.$$

Accurate to the additive constant, the last expression can be written in the form

$$E_k = -\sum_{(i,j)} U\sigma_i^2\sigma_j^2 - \sum_i f\sigma_i^2,$$

where $U = U_{11} + U_{22} - 2U_{12}$, $f = q(U_{12} - U_{22})$. We will call quantity U as the effective potential of the Coulomb interaction; magnetic atoms are attracted at $U > 0$ and repulsed at $U < 0$.

Taking into account that the number of magnetic atoms in the lattice is $\Sigma_i \sigma_i^2$, let us write the large statistic sum of the system as follows:

$$Z = \sum_{\{\sigma\}} \exp\left\{\left(\sum_{(i,j)}(J\sigma_i\sigma_j + U\sigma_i^2\sigma_j^2)\right.\right.$$
$$\left.\left. + (f+\mu)\sum_i \sigma_i^2 + H_e\sum_i \sigma_i\right)/kT\right\}, \quad (1)$$

where μ is the chemical potential and H_e is the external magnetic field, while summation is performed over all possible configurations $\{\sigma\}$.

Let us introduce quantities $p = \langle\sigma_i^2\rangle$ and $M = \langle\sigma_i\rangle/p$. It is clear that these quantities are independent of i since all lattice sites are equivalent (in the thermodynamic limit) and has a simple sense: p is the probability of the fact that the magnetic atom is arranged in this site (concentration), while M is its average spin.

To calculate M and p approximately, let us use the following procedure [7–9]. Let us determine local exchange field h_i and crystalline field φ_i of the ith site as $h_i = \Sigma\sigma_j$ and $\varphi_i = \Sigma\sigma_j^2$ (summation is performed over all sites neighboring to the ith sites). We will consider quantities h_i and φ_i as the values of random quantities h and φ with joint distribution function $W_1(h, \varphi)$. Then $\langle\sigma_i\rangle$ and $\langle\sigma_i^2\rangle$ average by ensemble are calculated as

$$pM = \left\langle \frac{\sinh(Kh+h_e)}{\cosh(Kh+h_e) + xe^{-L\varphi}} \right\rangle_{W_1(h,\varphi)}, \quad (2)$$

$$p = \left\langle \frac{\cosh(Kh+h_e)}{\cosh(Kh+h_e) + xe^{-L\varphi}} \right\rangle_{W_1(h,\varphi)}, \quad (3)$$

where $K = J/kT$, $L = U/kT$, $h_e = H_e/kT$, $x = \frac{1}{2}\exp(-(f+\mu)/kT)$ (k is the Boltzmann constant).

Let us now consider a cluster consisting of two neighboring sites and introduce exchange fields h_1 and

h_2 and crystalline fields φ_1 and φ_2 for each cluster site. Let us find average values of quantities $(\sigma_1 + \sigma_2)/2$ and $(\sigma_1^2 + \sigma_2^2)/2$ over the ensemble with a Hamiltonian

$$E_2(\sigma_1, \sigma_2) = -J\sigma_1\sigma_2 - Jh_1\sigma_1 - Jh_2\sigma_2$$
$$- U\sigma_1^2\sigma_2^2 - U\varphi_1\sigma_1^2 - U\varphi_2\sigma_2^2$$
$$- \mu(\sigma_1^2 + \sigma_2^2) - H_e(\sigma_1 + \sigma_2),$$

considering h_1, h_2, φ_1 and φ_2 as constants. Let us then introduce the joint distribution function $W_2(h_1, h_2, \varphi_1, \varphi_2)$ and average the result by this function

$$pM$$
$$= \left\langle \frac{\frac{1}{2}\Sigma_{\sigma_1,\sigma_2}(\sigma_1+\sigma_2)\exp(-E_2(\sigma_1,\sigma_2)/kT)}{\Sigma_{\sigma_1,\sigma_2}\exp(-E_2(\sigma_1,\sigma_2)/kT)} \right\rangle_{W_2(h_1,h_2,\varphi_1,\varphi_2)}, \quad (4)$$

$$p$$
$$= \left\langle \frac{\frac{1}{2}\Sigma_{\sigma_1,\sigma_2}(\sigma_1^2+\sigma_2^2)\exp(-E_2(\sigma_1,\sigma_2)/kT)}{\Sigma_{\sigma_1,\sigma_2}\exp(-E_2(\sigma_1,\sigma_2)/kT)} \right\rangle_{W_2(h_1,h_2,\varphi_1,\varphi_2)}. \quad (5)$$

Now, we can construct approximate methods of finding M and p in two ways. The first way is in that unknown distribution function $W_1(h, \varphi)$ in (2) and (3) (or function $W_2(h_1, h_2, \varphi_1, \varphi_2)$ in (4) and (5) is substituted by one or another approximate expression through desired values of M and p, due to which, (2) and (3) (or (4) and (5)) are transformed into equations relative to M and p. For example, if we take $W_1(h, \varphi) = \delta(h - qpM)\delta(\varphi - qp)$, i.e., substitute fields by their average values, we will derive the mean field method. On the other hand, if we construct the approximate expression for $W_1(h, \varphi)$ using a binomial distribution, we will derive the method described in [13]. Of course, we can also construct similar approximations for function $W_2(h_1, h_2, \varphi_1, \varphi_2)$.

The second method consists in the fact that both distribution functions $W_1(h, \varphi)$ and $W_2(h_1, h_2, \varphi_1, \varphi_2)$ are expressed through the same parameters m and ρ, equations for determining of which are obtained by leveling right sides (2) and (4) as well as (3) and (5). Such method of obtaining approximate equations can be found "renormalization-grouping" bearing in mind that the transition from a cluster with one site to a cluster with two sites can be considered as renormalization-grouping fixed-scale transformation [14]. As applied to the Ising magnet without impurities, one variant of this method is the known Bethe approximation [5]. Let us now construct the approximation similar to the Bethe approximation but for the Ising model with mobile nonmagnetic impurities.

3. BETHE APPROXIMATION

Taking into account the foregoing, we use the following approximations for field distribution functions:

$$W_1(h, \varphi) = \delta(h - q\rho m)\delta(\varphi - q\rho),$$

$$W_2(h_1, h_2, \varphi_1, \varphi_2) = \delta(h_1 - (q-1)\rho m)x$$
$$\times \delta(h_2 - (q-1)\rho m)\delta(\varphi_1 - (q-1)\rho)$$
$$\times \delta(\varphi_2 - (q-1)\rho),$$

where m and ρ are certain unknown parameters. Then we will derive from (2) and (3)

$$pM = \frac{\sinh(Kq\rho m + h_e)}{\cosh(Kq\rho m + h_e) + xe^{-Lq\rho}}, \quad (6)$$

$$p = \frac{\cosh(Kq\rho m + h_e)}{\cosh(Kq\rho m + h_e) + xe^{-Lq\rho}}, \quad (7)$$

and from (4) and (5)

$$pM = [\sinh(2K(q-1)\rho m + 2h_e)$$
$$+ 2xe^{-K-L-L(q-1)\rho}\sinh(K(q-1)\rho m + h_e)]/Z, \quad (8)$$

$$p = [\cosh(2K(q-1)\rho m + 2h_e)$$
$$+ 2xe^{-K-L-L(q-1)\rho}\cosh(K(q-1)\rho m + h_e) + e^{-2K}]/Z, \quad (9)$$

where

$$Z = \cosh(2K(q-1)\rho m + 2h_e)$$
$$+ 4xe^{-K-L-L(q-1)\rho}\cosh(K(q-1)\rho m + h_e)$$
$$+ 2x^2e^{-K-L-2L(q-1)\rho} + e^{-2K}.$$

Set of equations (6)−(9) at $h_e > 0$ (in this case, magnetization M is always positive) can be presented as follows. Let us introduce the notation

$$w = K(q-1)\rho m + h_e,$$

$$y = xe^{-K-L-L(q-1)\rho}, \quad \gamma = \frac{L}{K} = \frac{U}{J}.$$

Then we will derive from (6)−(9)

$$M = \tanh\left(\frac{qw - h_e}{q-1}\right), \quad (10)$$

$$p = \frac{2y\cosh(w) + \cosh(2w) + e^{-2K}}{2y^2e^{(1+\gamma)K} + 4y\cosh(w) + \cosh(2w) + e^{-2K}}, \quad (11)$$

$$\mu = -kT\ln\left(2y^qe^{q(1+\gamma)K}\left(\frac{p}{1-p}\right)^{q-1}(1-M^2)^{\frac{q-1}{2}}\right), \quad (12)$$

where

$$y = \frac{1}{2}(1 - e^{-2K})\frac{\sinh\left(\frac{qw - h_e}{q-1}\right)}{\sinh\left(\frac{w - h_e}{q-1}\right)} - \cosh(w). \quad (13)$$

In fact, these equations are parametric dependences $M(p, K, h_e)$ and $\mu(p, K, h_e)$ since, as we can easily convince, such range of values of parameter w always occurs at $h_e > 0$, for which p calculated by formula (11) runs the entire range from 0 to 1. Indeed, it follows from (11) and (13) at $w \longrightarrow h_e + 0$ that $y \longrightarrow +\infty$, while $p \longrightarrow 0$. On the other hand, when considering the asymptotic behavior of $y(w)$ (13) at $w \longrightarrow +\infty$, we can conclude that such value of w occurs, at which $y = 0$ and, according to (11), $p = 1$.

Let us now consider the case $h_2 = 0$. It turns out that the solution of set of equations (6)−(9) with $M \neq 0$ occurs only at $K > K_c(p)$, where $K_c(p)$ is determined as follows. Mutually equating quotients from dividing right parts of (6), (7) and (8), (9), we will derive

$$\frac{\sinh(2w) + 2y\sinh(w)}{\cosh(2w) + 2y\cosh(w) + e^{-2K}} = \tanh\left(\frac{qw}{q-1}\right), \quad (14)$$

where the value of y is expressed from (11) through p and w.

A positive root of Eq. (14) (which corresponds to the nonzero magnetization) disappears under the condition of equality of derivatives with respect to ω of the right and left parts of (14) at $w = 0$. This condition leads to the expression

$$p = \frac{(q-1)\tanh K_c}{e^{\gamma K_c}((q-1)\tanh K_c - 1)^2\cosh K_c + 2(q-1)\tanh K_c - 1}, \quad (15)$$

which determines $K_c(p)$. This result can be found by another method going to limit $w \longrightarrow 0$ in expressions (11) and (13) at $h_e = 0$.

Thus, for values of p from range $[p_c, 1]$ at $h_e = 0$ and $K > K_c(1) = \frac{1}{2}\ln\frac{q}{q-2}$, where

$$p_c = \frac{(q-1)\tanh K}{e^{\gamma K}((q-1)\tanh K - 1)^2\cosh K + 2(q-1)\tanh K - 1},$$

solution (6)–(9) can be found by formulas (10)–(13) from $h_e = 0$. For values of p from range $[0, p_c]$ or for all values of p at $K < K_c(1)$, magnetization $M = 0$, while the chemical potential

$$\mu = -kT\left(q\ln(A) + (q-1)\ln\frac{p}{1-p}\right), \qquad (16)$$

where

$$A = \frac{1}{2p}((1-2p)$$
$$+ \sqrt{(1-2p)^2 + 2p(1-p)(1 + e^{-2K})e^{(1+\gamma)K}}).$$

The calculation shows that chemical potential μ calculated according to formulas (12) and (16) is a monotonically increasing function of concentration p at all values of temperature only if $\gamma > -1$. If $\gamma \neq -1$, the value of K_k occurs, at which there is the concentration region for all $K > K_k$ with $\frac{\partial\mu}{\partial p} < 0$. This means that in addition to the magnetic phase transition in the system at $\gamma > -1$ (i.e., if the positive potential of the Coulomb interaction U is either positive or negative but does not exceed exchange interaction energy J by absolute value), the transition of the "liquid–gas" occurs, which leads to the formation of phases with various concentrations of magnetic atoms. To investigate the phase diagram of the system, Eqs. (6)–(9) should be added by the known Maxwell construction [6], according to which, the chemical potential with region $\frac{\partial\mu}{\partial p} < 0$ is replaced at a certain distance by a constant, which is determined by the rule of "equal
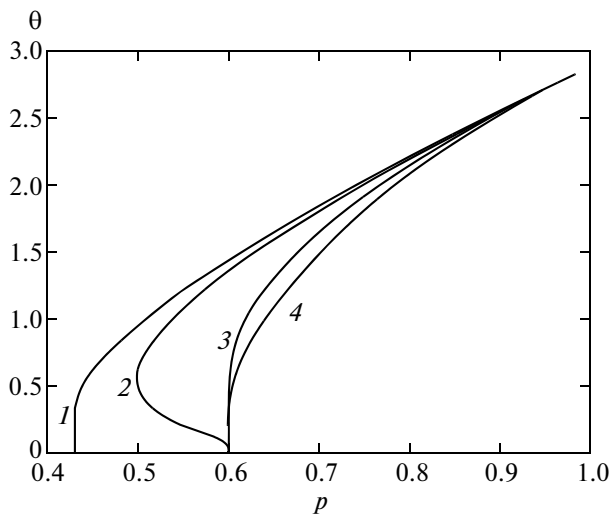
areas". However, we will not perform this investigation and limit ourselves by the analysis of the case $\gamma \leq -1$.

Let us initially investigate the dependence of critical temperature K_c^{-1} on the concentration of magnetic atoms p. This dependence is determined by Eq. (15). We will find from (15) at $p = 1$ that $K_c(1) = \frac{1}{2}\ln\frac{q}{q-2}$, which corresponds to the Curie temperature for pure Ising magnet in the Bethe approximation [5]. Function $K_c^{-1}(p)$ decreases monotonically as p decreases at $\gamma = -1$ and turns to zero at $p_{c1} = \frac{2(q-1)}{q^2 - 2}$ (curve 1 in Fig. 1). If $-3 < \gamma < -1$, functions $K_c^{-1}(p)$ are unambiguous and turn to zero at $p_{c2} = \frac{q-1}{2q-3}$ (curve 2 in Fig. 1). Functions $K_c^{-1}(p)$ at $\gamma \leq -3$ become monotonically decreasing with an increase in p turning to zero at p_{c2} (curve 3 in Fig. 1); at $\gamma \longrightarrow -\infty$, functions $K_c^{-1}(p)$ approach the limiting curve shown in Fig. 1 (curve 4).

We can find spontaneous magnetization M as a function of concentration p from Eqs. (10)–(13) at constant temperature $\theta = K^{-1}$. Particularly, function $M_0(p)$ at $\theta \longrightarrow 0$ can be interpreted as the probability of the fact that the randomly selected magnetic cluster belongs to an infinite cluster of such atoms [2]. If we come to limit $K \longrightarrow 0$ in formulas (10)–(13), we will

Fig. 1. Concentration dependences of the Curie temperature for $q = 4$. The concentration of magnetic atoms p is plotted along the abscissa, and the temperature parameter $\theta = kT/J$ is plotted along the ordinate. $\gamma = (1) -1$, $(2) -1.2$, $(3) -3$, and $(4) -\infty$.
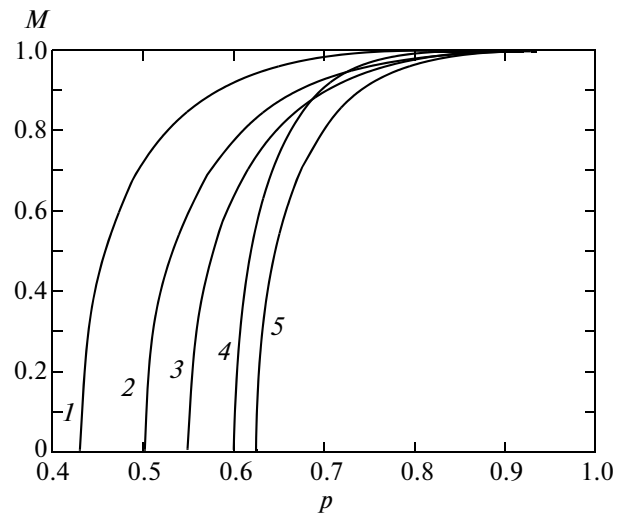

Fig. 2. Concentration dependences of the spontaneous magnetization for $q = 4$. The concentration of magnetic atoms p is plotted along the abscissa, and the spontaneous magnetization M is plotted along the ordinate. Curves 1 and 4 show limiting dependences $M(p)$ at $\theta = 0$: $(1) \gamma = -1$ and $(4) \gamma < -1$. Curves 2, 3, and 5 are constructed at $\theta = \frac{1}{3K_c(1)}$ for γ equal to -1, -1.5, and -3.5, respectively.

derive that $M(p) \longrightarrow M_{01}(p)$ at $\gamma = -1$, while $M(p) \longrightarrow M_{02}(p)$ at $\gamma < -1$. These curves are shown in Fig. 2 (curves *1* and *4*, respectively). They turn to zero at p equal to p_{c1} and p_{c2}, respectively. The spontaneous magnetization at any temperature θ is a monotonically increasing function of concentration p; this function depends on θ and γ at nonzero temperatures (Fig. 2, curves *2*, *3*, and *5*).

It follows from Fig. 1 that the dependence of magnetization M on temperature $\theta = K^{-1}$ will not be always monotonic. Indeed, the calculation shows that magnetization monotonically decreases as the temperature rises at $\gamma = -1$ and $\gamma \leq -3$ at any concentration p. If $-3 < \gamma < -1$, the temperature dependence of magnetization is more complex (Fig. 3). In this case, there is such concentration range of magnetic atoms, in which the spontaneous magnetization in the system appears only at a certain nonzero temperature θ_1, increases as the temperature increases passing through a maximum, and turns to zero again at temperature θ_2 (curve *1* in Fig. 3). Temperature θ_1 turns to zero at $p = p_{c2}$ (curve *2* in Fig. 3), while the spontaneous magnetization at $p > p_{c2}$ in the system also occurs at $\theta = 0$ as well (curve *3* in Fig. 3), although dependence $M(\theta)$ remains nonmonotonic as before.

4. ONE-DIMENSIONAL ISING MODEL WITH MOBILE IMPURITIES

It is known [4] that the Bethe approximation for the Ising magnet without impurities can be considered as an exact solution of the Ising problem for the Bethe lattice. The partial case of the Bethe lattice is a one-dimensional chain of magnetic atoms, for which an exact solution can be constructed using a transfer-matrix [5]. We can assume that the Bethe approximation considered in the previous section for the Ising magnet with mobile impurities considered in the previous section is also an exact solution of the problem for the Bethe lattice. To verify this assumption, let us calculate statistic sum (1) for a one-dimensional chain using a transfer matrix and compare the found result with solution (10)–(13) at $q = 2$.

Let us write the statistic sum of the one-dimensional chain, which consists of N sites in the following form:

$$Z_N$$
$$= \sum_{\{\sigma\}} \exp\left\{ \sum_{i=1}^{N} (K\sigma_i\sigma_{i+1} L\sigma_i^2\sigma_{i+1}^2 + r\sigma_i^2 + h_e\sigma_i) \right\}, \quad (17)$$

where $r = \mu/kT$ and cyclic boundary condition $\sigma_1 = \sigma_{N+1}$ is used. Calculating (17) in one or another manner, we can find the chemical potential and spontaneous magnetization from relationships

$$pM = \frac{1}{N}\frac{\partial \ln Z_N}{\partial h_e}, \quad p = \frac{1}{N}\frac{\partial \ln Z_N}{\partial r}. \quad (18)$$
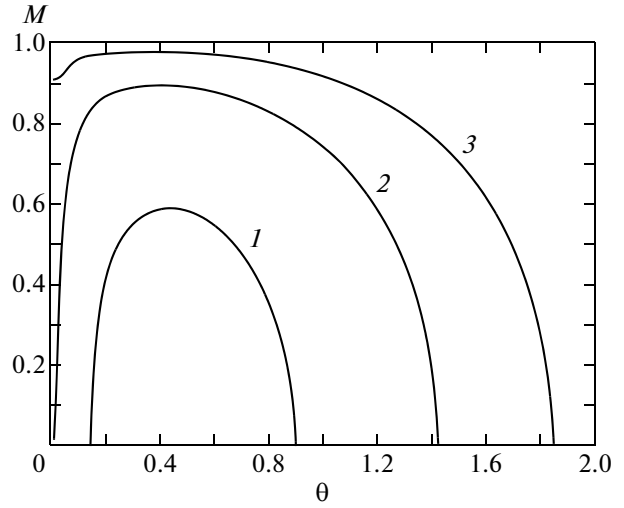
Fig. 3. Temperature dependences of the spontaneous magnetization at $\gamma = -1.1$ and $q = 4$. Temperature parameter $\theta = kT/J$ is plotted along the abscissa, and the spontaneous magnetization M is plotted along the ordinate. $p = $ (*1*) 0.5, (*2*) 0.6, and (*3*) 0.7.

To calculate Z_N, let us use the following method [5]. Let us consider transfer-matrix V

$$V = \begin{pmatrix} 1 & e^{(r+h_e)/2} & e^{(r-h_e)/2} \\ e^{(r+h_e)/2} & e^{K+L+r+h_e} & e^{-K+L+r} \\ e^{(r-h_e)/2} & e^{-K+L+r} & e^{K+L+r-h_e} \end{pmatrix}. \quad (19)$$

If λ_1, λ_2, and λ_3 are intrinsic numbers of matrix (19), then statistic sum (17) equals $Z_N = \lambda_1^N + \lambda_2^N + \lambda_3^N$. Let λ_1 is the maximal eigenvalue of matrix V. Then formulas (18) in the thermodynamic limit ($N \to \infty$) will transform to

$$pM = \frac{\partial \ln \lambda_1}{\partial h_e}, \quad p = \frac{\partial \ln \lambda_1}{\partial r}. \quad (20)$$

Eigenvalues (19) can be found from the characteristic equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0, \quad (21)$$

where

$$a = -(1 + 2e^{(1+\gamma)K+r}\cosh(h_e)),$$

$$b = 2(e^{2(\gamma K+r)}\sinh(2K) + (e^{(1+\gamma)K} - 1)e^r\cosh(h_e)),$$

$$c = (-4e^{\gamma K+2r}(e^{\gamma K}\cosh(K) - 1))\sinh(K).$$

Differentiating (21) with respect to r and h_e and using (20), we can write expressions for finding M and r through λ_1 and derivatives of coefficients a, b, and c

$$pM = -\frac{\lambda_1 \frac{\partial a}{\partial h_e} + \frac{\partial b}{\partial h_e}}{3\lambda_1^2 + 2a\lambda_1 + b},$$

$$p = -\frac{\lambda_1^2 \frac{\partial a}{\partial r} + \lambda_1 \frac{\partial b}{\partial r} + \frac{\partial c}{\partial r}}{(3\lambda_1^2 + 2a\lambda_1 + b)\lambda_1}. \qquad (22)$$

Solving (21) numerically or by Cardano's formulas [15] and finding λ_1, we can derive the dependence of magnetization M on p and h_e using (22). The calculation shows that thus calculated magnetization accurately coincides with that one calculated according to formulas (10)−(13) at $q = 2$.

5. CONCLUSIONS

Thus, based on our results, we can note the following.

Approximated self-consistent methods such as the Bethe approximation, which are applied to the Ising magnets without impurities or with frozen-in impurities [6−8], can be also used in the case of a more complex model—the Ising model with mobile impurities. The Bethe approximation applied to this model gives the following results.

(i) The states of the system are mainly determined by parameter γ—the ratio of the effective potential of the Coulomb interaction U and exchange interaction constant J. The separation into two macroscopic phases with different equilibrium concentrations of magnetic atoms is possible at $\gamma > -1$ in the system, while the system at $\gamma \leq -1$ remains homogeneous at all parameters.

(ii) If parameter $\gamma \leq -1$, i.e., when energy of the Coulomb repulsion of magnetic atoms is no smaller than the exchange interaction energy, there is their limiting concentration (similar to the percolation effect) below which the magnetization in the ground state is absent. This limiting value is $p_{c1} = \frac{2(q-1)}{q^2-2}$ at $\gamma = -1$ and $p_{c2} = \frac{q-1}{2q-3}$ at $\gamma < -1$.

(iii) There is the concentration region at $-3 < \gamma < -1$, in which the magnetization in the ground state is absent but appears as the temperature increases and vanishes again with the further increase in temperature (Figs. 1 and 3).

(iv) Our calculation shows that the Bethe approximation for the one-dimensional chain with mobile impurities is an exact solution of the problem as in the case of a pure Ising magnet. This allows us to assume that, for an arbitrary coordination number q, the Bethe approximation in a model with mobile impurities can be considered as an exact solution of the problem for the Bethe tree.

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